Synthetic Vector Analysis III
From Vector Analysis to Differential Forms

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Abstract

In our previous paper [International Journal of Theoretical Physics, **41** (2002), 1165-1190] we have shown, following the tradition of synthetic differential geometry, that div and rot are uniquely determined, so long as we require that the divergence theorem and the Stokes theorem should hold on the infinitesimal level. In this paper we will simplify the discussion considerably in terms of differential forms, leading to the natural derivation of exterior differentiation in the usual form.

1 Introduction

Vector analysis presupposes dogmatically that every physical quantity is either a scalar or a vector, excluding the possibility of tensors as natural physical quantities. In vector analysis, the force and the flux are equally vector fields, but, to tell the truth, the former is a field of tensors of degree 1, while the latter is a field of skew-symmetric tensors of degree 2. In electromagnetism, **E** and **B** are fields of tensors of degree 1, while **D** and **H** are fields of skew-symmetric tensors of degree 2. It is not desirable to apply div to **E** or **B**, though curl is indeed applicable to both of them. It is not desirable to apply curl to **D** or **H**, while div is indeed applicable to both of them. Since **E** and **D** as well as **B** and **H** are proportional in the vacuum, the confusion is apt to occur and develop ! Some physicists even insist wrongly that the CGS system of units, in which $\varepsilon_0 = \mu_0 = 1$ holds, is superior to the MKSA system of units.

Nowadays the number of textbooks on elementary physics (elementary electromagnetism in particular) using differential forms in place of vector analysis is increasing, though there are still only a few. It is easy to give a dictionary of vector analysis into the framework of differential forms, so that vector analysis is really to be absorbed into the calculus of differential forms. Nevertheless vector analysis is still popular among physicists and students of physics, mainly because vector analysis is highly intuitive, while the calculus of differential forms

is not. The exterior differentiation in the calculus of differential forms is usually given as a decree without taking care of its intuitive or physical foundations at all.

What is easily forgotten, such geniuses as Newton and Leibniz discussed advanced calculus in terms of nilpotent infinitesimals without using limits at all. It was in the 19th century, in the midst of the industrial revolution, that advanced calculus was reformulated in terms of limits, while nilpotent infinitesimals were intentionally neglected as anathema. Synthetic differential geometry, born in the middle of the 20th century, suceeded in reviving nilpotent infinitesimals in advanced calculus and differential geometry without hurting mathematical rigor at all. Newton and Leibniz saw nilpotent infinitesimals not in this world but in another world, and the 20th century witnessed powerful gadgets, such as seen in forcing techniques of set theory, sheaf theory and topos theory, by which many other mathematically meaningful worlds can be coherently constructed. Synthetic differential geometry regained the natural meaning of exterior differentiation in differential forms. The principal objective in this paper is to convince physicists that the exterior differentiation is well motivated, just as div and curl in vector analysis. The paper is more expository than anything else. We have tried to help physicists understand how naturally vector analysis develops into the calculus of differential forms.

2 Preliminaries

We assume that the reader is familiar with Chapter 1 of Lavendhomme [5]. The set $\mathbb{R}$ of (extended) real numbers is required to abide by the Kock-Lawvere axiom (cf. p.2 of [5]). We denote by D the set of real numbers whose squares vanish. The Kock-Lawvere axiom implies that, given a mapping $\varphi : \mathbb{R}^n \to \mathbb{R}$ and $\mathbf{x}, \mathbf{a} \in \mathbb{R}^n$, there exists a unique $\varphi'(\mathbf{x})(\mathbf{a}) \in \mathbb{R}^n$ such that

$$\varphi(\mathbf{x} + \mathbf{a}d) - \varphi(\mathbf{x}) = \varphi'(\mathbf{x})(\mathbf{a})d$$

for any $d \in D$. It can be shown easily that the mapping $\mathbf{a} \in \mathbb{R}^n \mapsto \varphi'(\mathbf{x})(\mathbf{a}) \in \mathbb{R}^n$, which is to be regarded as the derivative of φ at $\mathbf{x}$, is linear. The mapping $\varphi'(\mathbf{x})$ goes as follows:

$$\varphi'(\mathbf{x}) = \frac{\partial \varphi}{\partial x_1}(\mathbf{x})\mathbf{d}x_1 + ... + \frac{\partial \varphi}{\partial x_n}(\mathbf{x})\mathbf{d}x_n$$

We denote by $\mathbf{e}_i$

$$\begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

$(1 \leq i \leq n)$, where 1 is positioned at the i-th place. Given $\gamma : D^{m+1} \rightarrow \mathbb{R}^n$, $e \in D$ and a natural number i with $1 \leq i \leq m+1$, we write γ_e^i for the mapping $(d_1, ...d_m) \in D^m \mapsto \gamma(d_1, ..., d_{i-1}, e, d_i, ..., d_m) \in \mathbb{R}^n$.

Let us consider the usual three-dimensional space $\mathbb{R}^3$, which is the favorite space of vector analysis. Viewing the force $\mathbf{f}(\mathbf{x})$ at $\mathbf{x} \in\mathbb{R}^3$ as a vector in the usual way should be called an *idealistic* or *Platonic* view of force. Our *pragmatic* or *operational* view of force is to consider how to measure $\mathbf{f}(\mathbf{x})$ experimentally. If we move from $\mathbf{x}$ to $\mathbf{x} + \mathbf{a}d$ infinitesimally with $\mathbf{a} \in \mathbb{R}^3$ and $d \in D$, we get the power $\mathbf{f}(\mathbf{x}) \cdot \mathbf{a}d$, where $\cdot$ denotes the inner product of vectors. Our pragmatic view of force recommends that the force at $\mathbf{x}$ should not be $\mathbf{f}(\mathbf{x})$ but the linear mapping $\mathbf{a} \in \mathbb{R}^3 \mapsto \mathbf{f}(\mathbf{x}) \cdot \mathbf{a} \in\mathbb{R}$. We stress that $\mathbf{f}(\mathbf{x})$ is recognized via the linear mapping $\mathbf{a} \in \mathbb{R}^3 \mapsto \mathbf{f}(\mathbf{x}) \cdot \mathbf{a} \in\mathbb{R}$. This is our view of force as a tensor of degree 1. Therefore a field of forces is no other than a differential 1-form from a mathematical viewpoint.

Let us consider a flow of air in $\mathbb{R}^3$, which is very often represented by a field $\mathbf{f}$ of vectors. Our pragmatic view of flow recommends that we should measure how much air passes in a unit time through the infinitesimal parallelogram whose four vertices are $\mathbf{x}$, $\mathbf{x} + \mathbf{a}d_1$, $\mathbf{x} + \mathbf{b}d_2$ and $\mathbf{x} + \mathbf{a}d_1 + \mathbf{b}d_2$ with $\mathbf{x}, \mathbf{a}, \mathbf{b} \in\mathbb{R}^3$ and $d_1, d_2 \in D$. The result is surely $\mathbf{f}(\mathbf{x}) \cdot (\mathbf{a} \times \mathbf{b})d_1d_2$, where $\times$ stands for the vector product. We would like to consider pragamatically that the skew-symmetric bilinear mapping $(\mathbf{a}, \mathbf{b}) \in \mathbb{R}^3 \times \mathbb{R}^3 \mapsto \mathbf{f}(\mathbf{x}) \cdot (\mathbf{a} \times \mathbf{b}) \in \mathbb{R}$ is no other than the mathematical representation of the flow at $\mathbf{x}$. In this sense, the flow is represented by a field of skew-symmetric tensors of degree 2, namely, by a differential 2-form.

We know well that every linear mapping from $\mathbb{R}^3$ to $\mathbb{R}$ is of the form $\alpha_1\mathbf{d}x + \alpha_2\mathbf{d}y + \alpha_3\mathbf{d}z$ with $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$, while every skew-symmetric bilinear mapping from $\mathbb{R}^3 \times \mathbb{R}^3$ to $\mathbb{R}$ is of the form $\alpha_1\mathbf{d}y \wedge \mathbf{d}z + \alpha_2\mathbf{d}z \wedge \mathbf{d}x + \alpha_3\mathbf{d}x \wedge \mathbf{d}y$. We know well that every skew-symmetric trilinear mapping from $\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3$ to $\mathbb{R}$ is of the form $\alpha\mathbf{d}x \wedge \mathbf{d}y \wedge \mathbf{d}z$ with $\alpha \in \mathbb{R}$. More generally, every skew-symmetric k-linear mapping from $\underbrace{\mathbb{R}^n \times \cdots \times \mathbb{R}^n}_{k}$ to $\mathbb{R}$ is of the form

$$\sum_{1 \leq i_1 < ... < i_k \leq n} \alpha_{i_1,...,i_{k+1}}\mathbf{d}x_{i_1} \wedge ... \wedge \mathbf{d}x_{i_k}$$

In vector analysis, the operators div and rot are determined uniquely so that the divergence theorem and the Stokes' theorem should hold on the infinitesimal level respectively. In the same way, the exterior differentiation from a differential k-form to a differential $(k+1)$-form is determined uniquely so that Stokes' theorem should hold on the infinitesimal level. The principal objective in this paper is to give a lucid explanation on these facts as elementarily as possible from the standpoint of synthetic differential geometry, while avoiding the utmost generality, which would usually be liable to defy ordinary physicists.

3 The Fundamental Theorem for Gradient

Theorem 1 *Let φ be a scalar field on $\mathbb{R}^3$. Let $t : d \in D \mapsto \mathbf{x} + \mathbf{a}d$ be a tangent vector at $\mathbf{x}$ on $\mathbb{R}^3$. Let $e \in D$. Then we have*

$$\int_{\partial(t;e)} \varphi = \int_{(t;e)} \mathbf{d}\varphi$$

where

$$\partial(t;e) = (\mathbf{x} + \mathbf{a}e) - (\mathbf{x})$$

Proof. This is no other than the definition of $\mathbf{d}\varphi = \varphi'$, namely,

$$\varphi(\mathbf{x} + \mathbf{a}e) - \varphi(\mathbf{x}) = \varphi'(\mathbf{x})(\mathbf{a})e$$

■

4 The Fundamental Theorem for Rotation

Theorem 2 *Let $\omega = f\mathbf{d}x + g\mathbf{d}y + h\mathbf{d}z$ be a differential 1-form on $\mathbb{R}^3$. Let $\gamma : (d_1, d_2) \in D^2 \mapsto \mathbf{x} + \mathbf{a}d_1 + \mathbf{b}d_2$ be an infinitesimal parallelogram at $\mathbf{x}$ on $\mathbb{R}^3$. Let $(e_1, e_2) \in D^2$. Then we have*

$$\int_{\partial(\gamma;e_1,e_2)} \omega = \int_{(\gamma;e_1,e_2)} \mathbf{d}\omega$$

where

$$\begin{aligned}&\partial(\gamma;e_1,e_2)\\ &= (\gamma_0^2;e_1) + (\gamma_{e_1}^1;e_2) - (\gamma_{e_2}^2;e_1) - (\gamma_0^1;e_2)\end{aligned}$$

and

$$\mathbf{d}\omega = (\frac{\partial h}{\partial y} - \frac{\partial g}{\partial z})\mathbf{d}y \wedge \mathbf{d}z + (\frac{\partial f}{\partial z} - \frac{\partial h}{\partial x})\mathbf{d}z \wedge \mathbf{d}x + (\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y})\mathbf{d}x \wedge \mathbf{d}y$$

Proof. We have

$$\int_{\partial(\gamma;e_1,e_2)} \omega$$
$$= \int_{(\gamma_0^2;e_1)} \omega + \int_{(\gamma_{e_1}^1;e_2)} \omega - \int_{(\gamma_{e_2}^2;e_1)} \omega - \int_{(\gamma_0^1;e_2)} \omega$$
$$= \{f(\mathbf{x})a_1 + g(\mathbf{x})a_2 + h(\mathbf{x})a_3\}\, e_1$$
$$+ \{f(\mathbf{x}+\mathbf{a}e_1)b_1 + g(\mathbf{x}+\mathbf{a}e_1)b_2 + h(\mathbf{x}+\mathbf{a}e_1)b_3\}\, e_2$$
$$- \{f(\mathbf{x}+\mathbf{b}e_2)a_1 + g(\mathbf{x}+\mathbf{b}e_2)a_2 + h(\mathbf{x}+\mathbf{b}e_2)a_3\}\, e_1$$
$$- \{f(\mathbf{x})b_1 + g(\mathbf{x})b_2 + h(\mathbf{x})b_3\}\, e_2$$
$$= \{f'(\mathbf{x})(\mathbf{a})b_1 + g'(\mathbf{x})(\mathbf{a})b_2 + h'(\mathbf{x})(\mathbf{a})b_3\}\, e_1e_2$$
$$- \{f'(\mathbf{x})(\mathbf{b})a_1 + g'(\mathbf{x})(\mathbf{b})a_2 + h'(\mathbf{x})(\mathbf{b})a_3\}\, e_1e_2$$

[The first term delineated by $\{\}$ and followed by e_1e_2 is obtained by combining the second term and the fourth of the preceeding formula, while the second term delineated by $\{\}$ and followed by e_1e_2 is obtained by combining the first term and the third of the preceeding formula.]

$$= \left\{ \begin{pmatrix} f'(\mathbf{x})(\mathbf{a}) \\ g'(\mathbf{x})(\mathbf{a}) \\ h'(\mathbf{x})(\mathbf{a}) \end{pmatrix} \cdot \mathbf{b} - \begin{pmatrix} f'(\mathbf{x})(\mathbf{b}) \\ g'(\mathbf{x})(\mathbf{b}) \\ h'(\mathbf{x})(\mathbf{b}) \end{pmatrix} \cdot \mathbf{a} \right\} e_1e_2$$

Let $\varphi : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$ be the mapping

$$\varphi(\mathbf{a}, \mathbf{b}) = \begin{pmatrix} f'(\mathbf{x})(\mathbf{a}) \\ g'(\mathbf{x})(\mathbf{a}) \\ h'(\mathbf{x})(\mathbf{a}) \end{pmatrix} \cdot \mathbf{b} - \begin{pmatrix} f'(\mathbf{x})(\mathbf{b}) \\ g'(\mathbf{x})(\mathbf{b}) \\ h'(\mathbf{x})(\mathbf{b}) \end{pmatrix} \cdot \mathbf{a}$$

for any $(\mathbf{a}, \mathbf{b}) \in \mathbb{R}^3 \times \mathbb{R}^3$, so that

$$\int_{\partial(\gamma;e_1,e_2)} \omega = \varphi(\mathbf{a}, \mathbf{b})e_1e_2$$

Then it is easy to see that φ is a skew-symmetric bilinear mapping, so that φ is of the form

$$\varphi = \alpha_1 \mathbf{d}y \wedge \mathbf{d}z + \alpha_2 \mathbf{d}z \wedge \mathbf{d}x + \alpha_3 \mathbf{d}x \wedge \mathbf{d}y$$

with $\alpha_i \in \mathbb{R}$ $(i = 1, 2, 3)$. By taking

1. $\mathbf{a} = \mathbf{e}_2$ and $\mathbf{b} = \mathbf{e}_3$
2. $\mathbf{a} = \mathbf{e}_3$ and $\mathbf{b} = \mathbf{e}_1$, or
3. $\mathbf{a} = \mathbf{e}_1$ and $\mathbf{b} = \mathbf{e}_2$,

we get

$$\alpha_1 = \frac{\partial h}{\partial y}(\mathbf{x}) - \frac{\partial g}{\partial z}(\mathbf{x})$$
$$\alpha_2 = \frac{\partial f}{\partial z}(\mathbf{x}) - \frac{\partial h}{\partial x}(\mathbf{x})$$
$$\alpha_3 = \frac{\partial g}{\partial x}(\mathbf{x}) - \frac{\partial f}{\partial y}(\mathbf{x})$$

easily. This completes the proof. ■

5 The Fundamental Theorem for Divergence

Theorem 3 *Let $\omega = f\mathbf{d}y \wedge \mathbf{d}z \wedge + g\mathbf{d}z \wedge \mathbf{d}x + h\mathbf{d}x \wedge \mathbf{d}y$ be a differential 2-form on $\mathbb{R}^3$. Let $\gamma : (d_1, d_2, d_3) \in D^3 \mapsto \mathbf{x} + \mathbf{a}d_1 + \mathbf{b}d_2 + \mathbf{c}d_3$ be an infinitesimal parallelepiped at $\mathbf{x}$ on $\mathbb{R}^3$. Let $(e_1, e_2, e_3) \in D^3$. Then we have*

$$\int_{\partial(\gamma;e_1,e_2,e_3)} \omega = \int_{(\gamma;e_1,e_2,e_3)} \mathbf{d}\omega$$

where

$$\begin{aligned} &\partial(\gamma; e_1, e_2, e_3) \\ &= -(\gamma_0^1; e_2, e_3) + (\gamma_{e_1}^1; e_2, e_3) + (\gamma_0^2; e_1, e_3) - (\gamma_{e_2}^2; e_1, e_3) \\ &- (\gamma_0^3; e_1, e_2) + (\gamma_{e_3}^3; e_1, e_2) \end{aligned}$$

and

$$\mathbf{d}\omega = (\frac{\partial f}{\partial x} - \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z})\mathbf{d}x \wedge \mathbf{d}y \wedge \mathbf{d}z$$

Proof. We have

$$
\begin{aligned}
&\int_{\partial(\gamma;e_1,e_2,e_3)} \omega \\
&= -\int_{(\gamma_0^1;e_2,e_3)} \omega + \int_{(\gamma_{e_1}^1;e_2,e_3)} \omega + \int_{(\gamma_0^2;e_1,e_3)} \omega - \int_{(\gamma_{e_2}^2;e_1,e_3)} \omega - \int_{(\gamma_0^3;e_1,e_2)} \omega + \int_{(\gamma_{e_3}^3;e_1,e_2)} \omega \\
&= -\left\{ f(\mathbf{x}) \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} + g(\mathbf{x}) \begin{vmatrix} b_3 & c_3 \\ b_1 & c_1 \end{vmatrix} + h(\mathbf{x}) \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix} \right\} e_2e_3 \\
&+ \left\{ f(\mathbf{x}+\mathbf{a}e_1) \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} + g(\mathbf{x}+\mathbf{a}e_1) \begin{vmatrix} b_3 & c_3 \\ b_1 & c_1 \end{vmatrix} + h(\mathbf{x}+\mathbf{a}e_1) \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix} \right\} e_2e_3 \\
&+ \left\{ f(\mathbf{x}) \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + g(\mathbf{x}) \begin{vmatrix} a_3 & c_3 \\ a_1 & c_1 \end{vmatrix} + h(\mathbf{x}) \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix} \right\} e_1e_3 \\
&- \left\{ f(\mathbf{x}+\mathbf{b}e_1) \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + g(\mathbf{x}+\mathbf{b}e_1) \begin{vmatrix} a_3 & c_3 \\ a_1 & c_1 \end{vmatrix} + h(\mathbf{x}+\mathbf{b}e_1) \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix} \right\} e_1e_3 \\
&- \left\{ f(\mathbf{x}) \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix} + g(\mathbf{x}) \begin{vmatrix} a_3 & b_3 \\ a_1 & b_1 \end{vmatrix} + h(\mathbf{x}) \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \right\} e_1e_2 \\
&+ \left\{ f(\mathbf{x}+\mathbf{c}e_3) \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix} + g(\mathbf{x}+\mathbf{c}e_3) \begin{vmatrix} a_3 & b_3 \\ a_1 & b_1 \end{vmatrix} + h(\mathbf{x}+\mathbf{c}e_3) \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \right\} e_1e_2 \\
&= \left\{ f'(\mathbf{x})(\mathbf{a}) \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} + g'(\mathbf{x})(\mathbf{a}) \begin{vmatrix} b_3 & c_3 \\ b_1 & c_1 \end{vmatrix} + h'(\mathbf{x})(\mathbf{a}) \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix} \right\} e_1e_2e_3 \\
&- \left\{ f'(\mathbf{x})(\mathbf{b}) \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + g'(\mathbf{x})(\mathbf{b}) \begin{vmatrix} a_3 & c_3 \\ a_1 & c_1 \end{vmatrix} + h'(\mathbf{x})(\mathbf{b}) \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix} \right\} e_1e_2e_3 \\
&+ \left\{ f'(\mathbf{x})(\mathbf{c}) \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} + g'(\mathbf{x})(\mathbf{c}) \begin{vmatrix} b_3 & c_3 \\ b_1 & c_1 \end{vmatrix} + h'(\mathbf{x})(\mathbf{c}) \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix} \right\} e_1e_2e_3 \\
&= \left\{ \begin{vmatrix} f'(\mathbf{x})(\mathbf{a}) & & \\ g'(\mathbf{x})(\mathbf{a}) & \mathbf{b} & \mathbf{c} \\ h'(\mathbf{x})(\mathbf{a}) & & \end{vmatrix} + \begin{vmatrix} & f'(\mathbf{x})(\mathbf{b}) & \\ \mathbf{a} & g'(\mathbf{x})(\mathbf{b}) & \mathbf{c} \\ & h'(\mathbf{x})(\mathbf{b}) & \end{vmatrix} + \begin{vmatrix} & & f'(\mathbf{x})(\mathbf{c}) \\ \mathbf{a} & \mathbf{b} & g'(\mathbf{x})(\mathbf{c}) \\ & & h'(\mathbf{x})(\mathbf{c}) \end{vmatrix} \right\} e_1e_2e_3
\end{aligned}
$$

Let $\varphi : \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$ be the mapping

$$
\begin{aligned}
&\varphi(\mathbf{a},\mathbf{b},\mathbf{c}) \\
&= \begin{vmatrix} f'(\mathbf{x})(\mathbf{a}) & & \\ g'(\mathbf{x})(\mathbf{a}) & \mathbf{b} & \mathbf{c} \\ h'(\mathbf{x})(\mathbf{a}) & & \end{vmatrix} + \begin{vmatrix} & f'(\mathbf{x})(\mathbf{b}) & \\ \mathbf{a} & g'(\mathbf{x})(\mathbf{b}) & \mathbf{c} \\ & h'(\mathbf{x})(\mathbf{b}) & \end{vmatrix} + \begin{vmatrix} & & f'(\mathbf{x})(\mathbf{c}) \\ \mathbf{a} & \mathbf{b} & g'(\mathbf{x})(\mathbf{c}) \\ & & h'(\mathbf{x})(\mathbf{c}) \end{vmatrix}
\end{aligned}
$$

for any $(\mathbf{a},\mathbf{b},\mathbf{c}) \in \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3$, so that

$$\int_{\partial(\gamma;e_1,e_2,e_3)} \omega = \varphi(\mathbf{a},\mathbf{b},\mathbf{c})e_1e_2e_3$$

Then it is easy to see that φ is a skew-symmetric trilinear mapping, so that φ is of the form

$$\varphi = \alpha \mathbf{d}x \wedge \mathbf{d}y \wedge \mathbf{d}z$$

with $\alpha \in \mathbb{R}$. By taking $\mathbf{a} = \mathbf{e}_1$, $\mathbf{b} = \mathbf{e}_2$ and $\mathbf{c} = \mathbf{e}_3$, we get

$$\alpha = \frac{\partial f}{\partial x}(\mathbf{x}) + \frac{\partial g}{\partial y}(\mathbf{x}) + \frac{\partial h}{\partial z}(\mathbf{x})$$

easily. This completes the proof. ■

6 The Fundamental Theorem for Exterior Differentiation

Theorem 4 *Let $\omega = \sum_{1 \leq i_1 < ... < i_k \leq n} f_{i_1,...,i_k} \mathbf{d}x_{i_1} \wedge ... \wedge \mathbf{d}x_{i_k}$ be a differential k-form on $\mathbb{R}^n$. Let $\gamma : (d_1, ..., d_{k+1}) \in D^{k+1} \mapsto \mathbf{a}^1 d_1 + ... + \mathbf{a}^{k+1} d_{k+1}$ be an infinitesimal $(k+1)$-parallelepiped at $\mathbf{x}$ on $\mathbb{R}^n$. Let $(e_1, ..., e_{k+1}) \in D^{k+1}$. Then we have*

$$\int_{\partial(\gamma; e_1,...,e_{k+1})} \omega = \int_{(\gamma; e_1,...,e_{k+1})} \mathbf{d}\omega$$

where

$$\begin{aligned} &\partial(\gamma; e_1, ..., e_{k+1}) \\ &= \sum_{i=1}^{k+1} (-1)^i \left\{ (\gamma_0^i; e_1, ..., \widehat{e_i}, ..., e_{k+1}) - (\gamma_{e_i}^i; e_1, ..., \widehat{e_i}, ..., e_{k+1}) \right\} \end{aligned}$$

and

$$\mathbf{d}\omega = \sum_{1 \leq i_1 < ... < i_{k+1} \leq n} \left(\sum_{j=1}^{k+1} (-1)^{j+1} \frac{\partial f_{i_1,...,\widehat{i_j},...,i_{k+1}}}{\partial x_{i_j}} \right) \mathbf{d}x_{i_1} \wedge ... \wedge \mathbf{d}x_{i_{k+1}}$$

Proof. We have

$$
\begin{aligned}
&\int_{\partial(\gamma;e_1,\dots,e_{k+1})}\omega \\
&=\sum_{i=1}^{k+1}(-1)^i\left\{\int_{(\gamma_0^i;e_1,\dots,\widehat{e_i},\dots,e_{k+1})}\omega-\int_{(\gamma_{e_i}^i;e_1,\dots,\widehat{e_i},\dots,e_{k+1})}\omega\right\} \\
&=\sum_{1\le i_1<\dots<i_k\le n}\sum_{i=1}^{k+1}(-1)^i\left\{\begin{array}{c} f_{i_1,\dots,i_k}(\mathbf{x})\begin{vmatrix} a_{i_1}^1 & \cdots & a_{i_1}^{i-1} & a_{i_1}^{i+1} & \cdots & a_{i_1}^{k+1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{i_k}^1 & \cdots & a_{i_k}^{i-1} & a_{i_k}^{i+1} & \cdots & a_{i_k}^{k+1}\end{vmatrix} e_1\dots\widehat{e_i}\dots e_{k+1}- \\ f_{i_1,\dots,i_k}(\mathbf{x}+\mathbf{a}^ie_i)\begin{vmatrix} a_{i_1}^1 & \cdots & a_{i_1}^{i-1} & a_{i_1}^{i+1} & \cdots & a_{i_1}^{k+1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{i_k}^1 & \cdots & a_{i_k}^{i-1} & a_{i_k}^{i+1} & \cdots & a_{i_k}^{k+1}\end{vmatrix} e_1\dots\widehat{e_i}\dots e_{k+1}\end{array}\right\} \\
&=\sum_{1\le i_1<\dots<i_k\le n}\sum_{i=1}^{k+1}(-1)^{i+1}f'_{i_1,\dots,i_k}(\mathbf{x})(\mathbf{a}^i)\begin{vmatrix} a_{i_1}^1 & \cdots & a_{i_1}^{i-1} & a_{i_1}^{i+1} & \cdots & a_{i_1}^{k+1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{i_k}^1 & \cdots & a_{i_k}^{i-1} & a_{i_k}^{i+1} & \cdots & a_{i_k}^{k+1}\end{vmatrix} e_1\dots e_{k+1} \\
&=\sum_{1\le i_1<\dots<i_k\le n}\begin{vmatrix} f'_{i_1,\dots,i_k}(\mathbf{x})(\mathbf{a}^1) & \cdots & f'_{i_1,\dots,i_k}(\mathbf{x})(\mathbf{a}^{k+1}) \\ a_{i_1}^1 & \cdots & a_{i_1}^{k+1} \\ \vdots & \cdots & \vdots \\ a_{i_k}^1 & \cdots & a_{i_k}^{k+1}\end{vmatrix} e_1\dots e_{k+1}
\end{aligned}
$$

Let $\varphi:\underbrace{\mathbb{R}^n\times\dots\times\mathbb{R}^n}_{k+1}\to\mathbb{R}$ be the mapping

$$
\varphi(\mathbf{a}^1,\dots,\mathbf{a}^{k+1})=\sum_{1\le i_1<\dots<i_k\le n}\begin{vmatrix} f'_{i_1,\dots,i_k}(\mathbf{x})(\mathbf{a}^1) & \cdots & f'_{i_1,\dots,i_k}(\mathbf{x})(\mathbf{a}^{k+1}) \\ a_{i_1}^1 & \cdots & a_{i_1}^{k+1} \\ \vdots & \cdots & \vdots \\ a_{i_k}^1 & \cdots & a_{i_k}^{k+1}\end{vmatrix}
$$

for any $(\mathbf{a}^1,\dots,\mathbf{a}^{k+1})\in\underbrace{\mathbb{R}^n\times\dots\times\mathbb{R}^n}_{k+1}$, so that

$$
\int_{\partial(\gamma;e_1,\dots,e_{k+1})}\omega=\varphi(\mathbf{a}^1,\dots,\mathbf{a}^{k+1})e_1\dots e_{k+1}
$$

Then it is easy to see that φ is a skew-symmetric $(k+1)$-linear mapping, so that φ is of the form

$$
\varphi=\sum_{1\le i_1<\dots<i_{k+1}\le n}\alpha_{i_1,\dots,i_{k+1}}\mathbf{d}x_{i_1}\wedge\dots\wedge\mathbf{d}x_{i_{k+1}}
$$

with $\alpha_{i_1,\dots,i_{k+1}}\in\mathbb{R}$ $(1\le i_1<\dots<i_{k+1}\le n)$. By taking $\mathbf{a}^1=\mathbf{e}_{i_1},\mathbf{a}^2=$

$\mathbf{e}_{i_2}, ..., \mathbf{a}^{k+1} = \mathbf{e}_{i_{k+1}}$, we get

$$\alpha_{i_1,...,i_{k+1}} = \sum_{j=1}^{k+1} (-1)^{j+1} \frac{\partial f_{i_1,...,\widehat{i_j},...,i_{k+1}}}{\partial x_{i_j}}$$

easily. This completes the proof. ■

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